

Nonparametric Estimation in the Model of Moving Average.

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1 Introduction.

The subject of robust estimation in time series is widely discussed in literature. One of the approaches is to use GM-estimation. This method incorporates a broad class of nonparametric estimators which under suitable conditions includes estimators robust to outliers in data. For the linear models the sensitivity of GM-estimators to outliers have been studied in the work by Martin and Yohai [5], and influence functionals for this estimator were derived. In this paper we follow this direction and examine the asymptotical properties of the class of M-estimators, which is narrower than the class of GM-estimators, but gives more insight into asymptotical properties of such estimators. This paper gives an asymptotic expansion of the residual weighted empirical process, which allows to prove asymptotic normality of these estimators in case of non-smooth objective functions. For simplicity MA(1) model is considered, but it will be shown that even in this case mathematical techniques used to derive these asymptotic properties appear to be rather complicated. However, the approach used in this paper could be applied to GM-estimators and to more realistic models.

2 Main Results.

In this work we consider the model of moving average MA(1):

$$(1) \quad u_i = \varepsilon_i - \alpha \varepsilon_{i-1}, \quad i = 0, \pm 1, \pm 2, \dots,$$

where $\{\varepsilon_i\}$ - iid, $E\varepsilon_1 = 0$, $E\varepsilon_1^2 < \infty$, $|\alpha| < 1$.

Let u_1, \dots, u_n be the observations of a random variable u .

For every $\theta \in \mathbf{R}^1$ set

$$\varepsilon_0(\theta) = 0,$$

$$(2) \quad \varepsilon_i(\theta) = u_i + \theta \varepsilon_{i-1}(\theta), \quad i = 1, 2, \dots$$

It can easily be seen that

$$(3) \quad \varepsilon_i(\theta) = \sum_{j=0}^{i-1} \theta^j u_{i-j},$$

where $\varepsilon_i(\theta)$ are the residuals of the model (1). Assume for a moment that the equation of a moving average holds only for $i = 1, 2, \dots$ with $\varepsilon_0 = 0$. Also let's assume that variables $\{\varepsilon_i, i \geq 1\}$ have a density function $g(x)$. Then the maximum likelihood equation for the estimation of α can be constructed. Denote $\mathcal{U} = (u_1, \dots, u_n)^T$, $\mathcal{E} = (\varepsilon_1, \dots, \varepsilon_n)^T$, and the matrix $J(\alpha)$ as

$$J(\alpha) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \alpha & 1 & 0 & \dots & 0 \\ \alpha^2 & \alpha & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha^{n-1} & \alpha^{n-2} & \alpha^{n-3} & \dots & 1 \end{pmatrix}.$$

Then the equations $\varepsilon_i = \sum_{j=0}^{i-1} \alpha^j u_{i-j}$, $i = 1, \dots, n$ can be rewritten as:

$$\mathcal{E} = J(\alpha)\mathcal{U}$$

If $f_{\mathcal{U}}(y_1, \dots, y_n)$ is the density function of the vector \mathcal{U} , then

$$f_{\mathcal{U}}(y_1, \dots, y_n) = \prod_{i=1}^n g\left(\sum_{j=0}^{i-1} \alpha^j y_{i-j}\right).$$

Therefore, the maximum likelihood estimator, which is defined as a solution of the maximization problem

$$\log f_{\mathcal{U}}(y_1, \dots, y_n) \longrightarrow \sup_{\theta},$$

can be obtained as a root of the equation

$$\sum_{i=1}^n \frac{\partial \varepsilon_i(\theta)}{\partial \theta} \frac{g'(\varepsilon_i(\theta))}{g(\varepsilon_i(\theta))} = 0,$$

In this paper we examine a natural generalization of this estimator for the model(1), namely, M -estimator $\hat{\alpha}_n$ of the parameter α .

The estimator $\hat{\alpha}_n$ is defined as a solution of the equation

$$\frac{1}{n} \sum_{i=1}^n \frac{\partial \varepsilon_i(\theta)}{\partial \theta} \Psi(\varepsilon_i(\theta)) = 0,$$

with $\varepsilon_i(\theta)$, determined by (2) or (3), where $\Psi(x)$ is an a priori known function, which we will choose later.

The asymptotic distribution of the estimator $\hat{\alpha}_n$ in the case when the function $\Psi(\cdot)$ satisfies $\text{Var} |_{-\infty}^{+\infty}[\Psi] < \infty$ will be derived. This result will be obtained with the help of asymptotically uniform expansion of the residual weighted empirical process, which will be defined later and which will be of interest by its own.

We fix some notation:

$$\begin{aligned} l_n(\theta) &:= \frac{1}{n} \sum_{k=1}^n \frac{\partial \varepsilon_k(\theta)}{\partial \theta} \Psi(\varepsilon_k(\theta)), & \tilde{l}_n(\alpha) &:= \frac{1}{n} \sum_{k=1}^n \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \Psi(\varepsilon_k), \\ u_n(x, \theta) &:= \frac{1}{n} \sum_{k=1}^n \frac{\partial \varepsilon_k(\theta)}{\partial \theta} \mathbf{I}(\varepsilon_k(\theta) \leq x), & \tilde{u}_n(x, \alpha) &:= \frac{1}{n} \sum_{k=1}^n \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \mathbf{I}(\varepsilon_k \leq x), \end{aligned}$$

where ε_i are derived from (1):

$$\varepsilon_i = \sum_{j \geq 0} \alpha^j u_{i-j}.$$

If the variation of Ψ is bounded, the following is true:

$$\begin{aligned} n^{1/2} \left[l_n(\alpha + n^{-1/2}\tau) - \tilde{l}_n(\alpha) \right] &= \int_{-\infty}^{+\infty} \Psi(x) dn^{1/2} [u_n(x, \alpha + n^{-1/2}\tau) - \tilde{u}_n(x, \alpha)] = \\ &= - \int_{-\infty}^{+\infty} n^{1/2} [u_n(x, \alpha + n^{-1/2}\tau) - \tilde{u}_n(x, \alpha)] d\Psi(x) + \left\{ n^{1/2} [u_n(x, \alpha + n^{-1/2}\tau) - \tilde{u}_n(x, \alpha)] \Psi(x) \right\}_{-\infty}^{+\infty}. \end{aligned}$$

It will be proved below that the second term is equal to zero. Now consider the first term.

Theorem 1. *Assume that the following conditions hold:*

$$E(\varepsilon_1)^8 < \infty;$$

$$g(x) > 0, \quad \lim_{x \rightarrow \infty} g(x) = 0, \quad \sup_{x \in \mathbf{R}} |g'(x)| < \infty.$$

Then

$$\sup_{x \in \mathbf{R}, |\tau| < \theta} \left| n^{1/2} [u_n(x, \alpha + n^{-1/2}\tau) - \tilde{u}_n(x, \alpha)] + \tau g(x) \frac{E\varepsilon_1^2}{1 - \alpha^2} \right| = o_P(1).$$

The first theorem will be proved in section 2. The next theorem is the main result of the work and its proof utilizes the first theorem.

Theorem 2. *Assume that the following conditions hold:*

$$(i) \quad E(\varepsilon_1)^8 < \infty,$$

$$g(x) > 0, \quad \lim_{x \rightarrow \infty} g(x) = 0, \quad \lim_{x \rightarrow -\infty} g(x) = 0, \quad \sup_{x \in \mathbf{R}} |g'(x)| < \infty;$$

$$(ii) \quad \text{Var} \left[\int_{-\infty}^{+\infty} \Psi \right] < \infty,$$

$$\int_{-\infty}^{\infty} g d\Psi \neq 0,$$

$$E[\Psi(\varepsilon_1)] = 0.$$

then

1) if Ψ is continuous on \mathbf{R}^1 , then with probability tending to one, there exists a $n^{1/2}$ -consistent solution of the following equation

$$(4) \quad n^{-1} \sum_{k=1}^n \frac{\partial \varepsilon_k(\theta)}{\partial \theta} \Psi(\varepsilon_k(\theta)) = 0;$$

2) for any $n^{1/2}$ - consistent solution $\hat{\alpha}_n$ of (4) the following statement holds

$$n^{1/2}(\hat{\alpha}_n - \alpha) \xrightarrow{d} N(0, \sigma_{\Psi}^2(\alpha)),$$

where

$$\sigma_{\Psi}^2(\alpha) = (1 - \alpha^2) \frac{E\Psi^2(\varepsilon_1)}{\left(\int_{-\infty}^{+\infty} g d\Psi \right)^2 E(\varepsilon_1)^2}.$$

The function $\Psi(x) = F(x) - \frac{1}{2}$ where $F(x)$ is a continuous distribution function of a certain zero mean symmetrical distribution satisfies the conditions of theorem 2 in case ε_i are symmetrical zero mean random variables.

3 Proofs of the theorems

Lemma 1. *If for some $p \geq 1$ $E|\varepsilon_1|^p < \infty$, then:*

$$\sup_{1 \leq k \leq n} E \left| \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \right|^p < \infty;$$

$$\sup_{1 \leq k \leq n} E \left| \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \right|^p < \infty.$$

Proof.

$$\frac{\partial \varepsilon_k(\alpha)}{\partial \theta} = \sum_{j=1}^{k-1} j \alpha^{j-1} u_{k-1}.$$

$$E \left| \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \right|^p \leq E \left(\sum_{j=1}^{k-1} j |\alpha|^{j-1} |u_{k-1}| \right)^p \leq c E \left(\sum_{j=1}^{k-1} \alpha_1^j |u_{k-1}| \right)^p \leq$$

$$\begin{aligned}
&\leq c\mathbb{E} \left(\sum_{j=1}^{k-1} \alpha_1^j |\varepsilon_{k-1}| + |\alpha| \sum_{j=1}^{k-1} \alpha_1^j |\varepsilon_{k-1-j}| \right)^p \leq \\
&\leq 2c \left[\mathbb{E} \left(\sum_{j=1}^{k-1} \alpha_1^j |\varepsilon_{k-1}| \right)^p + |\alpha| \mathbb{E} \left(\sum_{j=1}^{k-1} \alpha_1^j |\varepsilon_{k-1-j}| \right)^p \right],
\end{aligned}$$

where $\alpha_1 \in [0, +\infty)$: $\begin{cases} |\alpha| < \alpha_1 < 1 \\ j|\alpha|^{j-1} < c\alpha_1^j \end{cases}$. Therefore, it is sufficient to prove

$$\sup_{1 \leq k \leq n} \left\{ \mathbb{E} \left(\sum_{j=1}^{k-1} \alpha_1^j |\varepsilon_{k-j}| \right)^p \right\}^{1/p} < \infty.$$

This clear from the Minkovsky's inequality:

$$\begin{aligned}
\left\{ \mathbb{E} \left(\sum_{j=1}^{k-1} \alpha_1^j |\varepsilon_{k-j}| \right)^p \right\}^{1/p} &\leq \alpha_1 \{ \mathbb{E}(\varepsilon_1)^p \}^{1/p} + \left\{ \mathbb{E} \left(\sum_{j=2}^{k-1} \alpha_1^j |\varepsilon_{k-j}| \right)^p \right\}^{1/p} \leq \dots \leq \\
&\leq (\alpha_1 + \dots + \alpha_1^{k-1}) \{ \mathbb{E}|\varepsilon_1|^p \}^{1/p} < \infty,
\end{aligned}$$

what proves the first claim of the lemma. The second one can be proved in the similar way.

Lemma 2. *set*

$$\sigma_k(\tau) := -\tau n^{-1/2} \sum_{t \geq k} \alpha^t \varepsilon_{k-1-t} + \tau n^{-1/2} \sum_{t=0}^{k-1} \left[(\alpha + \tau n^{-1/2})^t - \alpha^t \right] \varepsilon_{k-1-t} - \left(\alpha_n + \tau n^{-1/2} \right)^k \varepsilon_0$$

if $E(\varepsilon_1)^4 < \infty$, then there exists such a $\hat{\sigma}_k$, that

$$\sup_{|\tau| \leq \theta} |\sigma_k(\tau)| \leq \hat{\sigma}_k,$$

$$\sup_{1 \leq k \leq n} E(\hat{\sigma}_k)^4 < \infty.$$

Proof. Let $B \in (0, 1)$: $|\alpha| + \theta n^{-1/2} < B$ for $n > n_0, 0 < \theta < \infty$, then $\sigma_k(\tau) \leq \hat{\sigma}_k$, where

$$\hat{\sigma}_k := \theta n^{-1/2} \sum_{t \geq k} B^t |\varepsilon_{k-1-t}| + \theta^2 n^{-1} \sum_{t \geq 1} t B^{t-1} |\varepsilon_{k-1-t}| - B^k \varepsilon_0.$$

$$E(\sigma_k)^4 = E \left(\theta n^{-1/2} \sum_{t \geq k} B^t |\varepsilon_{k-1-t}| + \theta^2 n^{-1} \sum_{t \geq 1} t B^{t-1} |\varepsilon_{k-1-t}| - B^k \varepsilon_0 \right)^4.$$

For $\forall a, b > 0, p \geq 1$ we have the following inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$. Hence, it follows that

$$E(\sigma_k)^4 \leq 2^3 \cdot \left[E \left(\theta n^{-1/2} \sum_{t \geq k} B^t |\varepsilon_{k-1-t}| \right)^4 + E \left(\theta^2 n^{-1} \sum_{t \geq 1} t B^{t-1} |\varepsilon_{k-1-t}| - B^k \varepsilon_0 \right)^4 \right].$$

By Minkovsky's inequality:

$$\begin{aligned} & \left\{ E \left(\theta n^{-1/2} \sum_{t \geq k} B^t |\varepsilon_{k-1-t}| \right)^4 \right\}^{1/4} \leq \left\{ \theta^4 n^{-2} B^{4k} E(\varepsilon_1)^4 \right\}^{1/4} + \\ & + \left\{ E \left(\theta n^{-1/2} \sum_{t \geq k+1} B^t |\varepsilon_{k-1-t}| \right)^4 \right\}^{1/4} \leq \left\{ \theta^4 n^{-2} (B^{4k} + B^{4(k+1)} + \dots) E(\varepsilon_1)^4 \right\}^{1/4} < \infty, \end{aligned}$$

hence,

$$\begin{aligned} & E \left(\theta n^{-1/2} \sum_{t \geq k} B^t |\varepsilon_{k-1-t}| \right)^4 < \infty. \\ & \left\{ E \left(\theta^2 n^{-1} \sum_{t \geq 1} t B^{t-1} |\varepsilon_{k-1-t}| - B^k \varepsilon_0 \right)^4 \right\}^{1/4} \leq \\ & \leq \left\{ \theta^8 n^{-4} (1 \cdot B^0 + 2 \cdot B^1 + \dots + t B^{t-1} + \dots) E(\varepsilon_1)^4 - B^k E(\varepsilon_1)^4 \right\}^{1/4}. \end{aligned}$$

Since the series $\sum_{t \geq 1} t B^{t-1}$ converges, it follows that

$$E \left(\theta^2 n^{-1} \sum_{t \geq 1} t B^{t-1} |\varepsilon_{k-1-t}| - B^k \varepsilon_0 \right)^4 < \infty,$$

what proves the lemma.

Lemma 3. *Let the following conditions hold:*

$$(i) \quad E(\varepsilon_1)^8 < \infty$$

(ii) $g(x)$ is the density function of ε_i that satisfies the following conditions:

$$\lim_{x \rightarrow \infty} g(x) = 0,$$

$$\sup_x |g'(x)| < \infty.$$

For $z_{1n}(x, \tau)$ defined as

$$z_{1n}(x, \tau) := n^{-1/2} \sum_{k=1}^n \left(\frac{\partial \varepsilon_k(\alpha + n^{-1/2}\tau)}{\partial \theta} - \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \right) \mathbf{I} \left(\varepsilon_k(\alpha + n^{-1/2}\tau) \leq x \right).$$

the following holds:

$$|z_{1n}(x, \tau)| = o_{\mathbf{P}}(1).$$

Proof.

$$\begin{aligned} z_{1n}(x, \tau) &= n^{-1/2} \sum_{k=1}^n \left(\frac{\partial \varepsilon_k(\alpha + n^{-1/2}\tau)}{\partial \theta} - \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \right) \mathbf{I} \left(\varepsilon_k(\alpha + n^{-1/2}\tau) \leq x \right) = \\ &= n^{-1/2} \sum_{k=1}^n \tau n^{-1/2} \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \mathbf{I} \left(\varepsilon_k(\alpha + n^{-1/2}\tau) \leq x \right) + \\ &\quad + n^{-1/2} \sum_{k=1}^n \tau^2 n^{-1} \frac{\partial^3 \varepsilon_k(\xi)}{\partial \theta^3} \mathbf{I} \left(\varepsilon_k(\alpha + n^{-1/2}\tau) \leq x \right) = \\ &= \tau n^{-1} \sum_{k=1}^n \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \mathbf{I} \left(\varepsilon_k(\alpha + n^{-1/2}\tau) \leq x \right) + o_{\mathbf{P}}(1), \end{aligned}$$

where $\xi \in [\alpha, \alpha + n^{-1/2}\tau]$. Therefore

$$\begin{aligned} |z_{1n}(x, \tau)| &\leq \left| \tau n^{-1} \sum_{k=1}^n \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \left[\mathbf{I} \left(\varepsilon_k(\alpha + n^{-1/2}\tau) \leq x \right) - \mathbf{I}(\varepsilon_k \leq x) \right] \right| + \\ &\quad + \left| \tau n^{-1} \sum_{k=1}^n \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \mathbf{I}(\varepsilon_k \leq x) \right| + o_{\mathbf{P}}(1). \end{aligned}$$

To transform the right hand of this inequality, we introduce the random process $L_n^+(x, \tau)$:

$$L_n^+(x, \tau) := \tau n^{-1} \sum_{k=1}^n a_k^+ \left[\mathbf{I} \left(\varepsilon_k(\alpha + n^{-1/2}\tau) \leq x \right) - \mathbf{I}(\varepsilon_k \leq x) \right],$$

where

$$a_k^+ := \begin{cases} \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2}, & \text{if } \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} > 0, \\ 0, & \text{if } \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \leq 0. \end{cases}$$

$$L_n^-(x, \tau) := \tau n^{-1} \sum_{k=1}^n a_k^- \left[\mathbf{I}(\varepsilon_k(\alpha + n^{-1/2} \tau) \leq x) - \mathbf{I}(\varepsilon_k \leq x) \right],$$

$$a_k^- := \begin{cases} -\frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2}, & \text{if } \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} < 0, \\ 0, & \text{if } \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \geq 0. \end{cases}$$

$$\left| \tau n^{-1} \sum_{k=1}^n \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \left[\mathbf{I}(\varepsilon_k(\alpha + n^{-1/2} \tau) \leq x) - \mathbf{I}(\varepsilon_k \leq x) \right] \right| \leq |L_n^+(x, \tau)| + |L_n^-(x, \tau)|.$$

We first show that $L_n^+(x, \tau) = o_P(1)$. The proof utilizes the expansion obtained in lemma 7.1 [1]

$$\varepsilon_k(\alpha + \tau n^{-1/2}) = \varepsilon_k + \tau n^{-1/2} \mu_{k-1} + \sigma_k(\tau),$$

and lemma 2.

$$\begin{aligned} |L_n^+(x, \tau)| &= \left| \tau n^{-1} \sum_{k=1}^n a_k^+(\alpha) \left[\mathbf{I}(\varepsilon_k \leq x - \tau n^{-1/2} \mu_{k-1} - \sigma_k(\tau)) - \mathbf{I}(\varepsilon_k \leq x) \right] \right| \leq \\ &\leq \left| \tau n^{-1} \sum_{k=1}^n a_k^+(\alpha) \left[\mathbf{I}(\varepsilon_k \leq x - \tau n^{-1/2} \mu_{k-1} + \hat{\sigma}_k) - \mathbf{I}(\varepsilon_k \leq x) \right] \right|. \end{aligned}$$

Consider random processes $v_n(x, \tau), \tilde{v}_n(x)$:

$$v_n(x, \tau) := n^{-1} \sum_{k=1}^n a_k^+ \left[\mathbf{I}(\varepsilon_k \leq x - \tau n^{-1/2} \mu_{k-1} + \hat{\sigma}_k) - G(x - \tau n^{-1/2} \mu_{k-1} + \hat{\sigma}_k) \right],$$

$$\tilde{v}_n(x) := n^{-1} \sum_{k=1}^n a_k^+ [\mathbf{I}(\varepsilon_k \leq x) - G(x)].$$

then

$$\begin{aligned} &\left| \tau n^{-1} \sum_{k=1}^n a_k^+(\alpha) \left[\mathbf{I}(\varepsilon_k \leq x - \tau n^{-1/2} \mu_{k-1} + \hat{\sigma}_k) - \mathbf{I}(\varepsilon_k \leq x) \right] \right| \leq \\ &\leq |\tau(v_n(x, \tau) - \tilde{v}_n(x))| + \left| n^{-1} \sum_{k=1}^n a_k^+(\alpha) \left(G(x - \tau n^{-1/2} \mu_{k-1} + \hat{\sigma}_k) - G(x) \right) \right|, \end{aligned}$$

and

$$\begin{aligned} & \left| n^{-1} \sum_{k=1}^n a_k^+(\alpha) \left(G \left(x - \tau n^{-1/2} \mu_{k-1} + \hat{\sigma}_k \right) - G(x) \right) \right| = \\ & = \left| n^{-3/2} \sum_{k=1}^n a_k^+(\alpha) \tau \mu_{k-1} g(\xi) + n^{-1} \sum_{k=1}^n a_k^+(\alpha) \hat{\sigma}_k g(\xi) + o_P(1) \right| = o_P(1), \end{aligned}$$

where $\xi \in [x - \tau n^{-1/2} \mu_{k-1} + \hat{\sigma}_k, x]$. To prove that $|v_n(x, \tau) - \tilde{v}_n(x)| = o_P(1)$ we use theorem 2.1 from [2]. Let's check whether the conditions of this theorem . The condition $E|\varepsilon_1|^8 < \infty$ implies $E|\varepsilon_1|^4 < \infty$, and because of the lemma1 the latter yields

$$\sup_{1 \leq k \leq n} E \left(\frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \right)^4 < \infty.$$

Hence,

$$n^{-1} \sum_{k=1}^n E (a_k^+(\alpha))^4 = O(1).$$

To check the condition $n^{-1} \sum_{k=1}^n E \left[(a_k^+(\alpha))^4 \hat{\sigma}_k^2 \right] = O(1)$ we use Cauchy-Bunyakovskii inequality:

$$n^{-1} \sum_{k=1}^n E \left[(a_k^+(\alpha))^4 \hat{\sigma}_k^2 \right] \leq n^{-1} \sum_{k=1}^n \left[E (a_k^+(\alpha))^8 \right]^{1/2} [E \hat{\sigma}_k^4]^{1/2}.$$

because of the lemma1 and lemma2:

$$\sup_{1 \leq k \leq n} E (a_k^+(\alpha))^8 < \infty,$$

$$\sup_{1 \leq k \leq n} E \hat{\sigma}_k^4 < \infty,$$

therefore,

$$n^{-1} \sum_{k=1}^n E \left[(a_k^+(\alpha))^4 \hat{\sigma}_k^2 \right] = O(1).$$

Now we check the following condition:

$$n^{-1} \sum_{k=1}^n a_k^+(\alpha) \hat{\sigma}_k^2 = O_P(1).$$

Lemma 2 yields $\sup_{1 \leq k \leq n} E\hat{\sigma}_k^2 < \infty$,

$$n^{-1} \sum_{k=1}^n a_k^+(\alpha) \hat{\sigma}_k^2 \leq \left(\sup_{1 \leq k \leq n} \hat{\sigma}_k^2 \right) n^{-1} \sum_{k=1}^n a_k^+(\alpha) = \left(\sup_{1 \leq k \leq n} \hat{\sigma}_k^2 \right) n^{-1} \sum_{k=1}^n \left| \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \right|.$$

Consider a random process

$$\frac{\partial^2 \tilde{\varepsilon}(\alpha)}{\partial \theta^2} = \sum_{j=2}^{\infty} (j(j-1)|\alpha|^{j-2}|u_{-j}|).$$

then

$$\begin{aligned} E \left(\frac{\partial^2 \tilde{\varepsilon}(\alpha)}{\partial \theta^2} - \left| \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \right| \right)^2 &\leq E \left(\sum_{j=k+1}^{\infty} j(j-1)|\alpha|^{j-2}|u_{-j}| \right)^2 \leq E \left(\sum_{j=k+1}^{\infty} c\alpha^j |u_{-j}| \right)^2 \leq \\ &\leq 2c^2 \left[E \left(\sum_{j=k+1}^{\infty} \alpha_1^j |\varepsilon_{-j}| \right)^2 + E \left(|\alpha| \sum_{j=k+1}^{\infty} \alpha_1^j |\varepsilon_{-j-1}| \right)^2 \right] = \\ &= 2c^2 E \varepsilon_1^2 \left[\sum_{j=k+1}^{\infty} \alpha_1^{2j} + |\alpha| \sum_{j=k+1}^{\infty} \alpha_1^{2j} \right] \longrightarrow 0, k \rightarrow \infty, \end{aligned}$$

where $\alpha_1 : \begin{cases} |\alpha| < \alpha_1 < 1 \\ j(j-1)|\alpha|^{j-2} \leq c\alpha_1^j \end{cases}$.

Hence

$$n^{-1} \sum_{k=1}^n \left| \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \right| \xrightarrow{P} E \left(\sum_{j=2}^{\infty} j(j-1)|\alpha|^{j-2}|u_{-j}| \right)^2,$$

and

$$n^{-1} \sum_{k=1}^n a_k^+(\alpha) \hat{\sigma}_k^2 = O_P(1).$$

So all the conditions of the theorem 2.1 from [2] hold, therefore

$$\sup_{x \in \mathbf{R}^1, |\tau| \leq \theta} n^{1/2} [v_n(x, \tau) - \tilde{v}_n(x)] = o_P(1).$$

we have proved that

$$\left| \tau n^{-1} \sum_{k=1}^n a_k^+(\alpha) \left[I \left(\varepsilon_k \leq x - \tau n^{-1/2} \mu_{k-1} + \hat{\sigma}_k \right) - I(\varepsilon_k \leq x) \right] \right| = o_P(1),$$

and

$$|L_n^+(x, \tau)| = o_P(1).$$

Therefore,

$$\left| \tau n^{-1} \sum_{k=1}^n \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \left[\mathbf{I}(\varepsilon_k(\alpha + n^{-1/2} \tau) \leq x) - \mathbf{I}(\varepsilon_k \leq x) \right] \right| = o_P(1).$$

To finish the proof of the lemma we need to show that

$$\left| \tau n^{-1} \sum_{k=1}^n \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \mathbf{I}(\varepsilon_k \leq x) \right| = o_P(1).$$

$$\left| \tau n^{-1} \sum_{k=1}^n \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \mathbf{I}(\varepsilon_k \leq x) \right| \leq \left| \tau n^{-1} \sum_{k=1}^n \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} [\mathbf{I}(\varepsilon_k \leq x) - G(x)] \right| + \left| \tau n^{-1} G(x) \sum_{k=1}^n \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \right|.$$

We apply theorem 2.1 from [2] to process $n^{-1} \sum_{k=1}^n \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} [\mathbf{I}(\varepsilon_k \leq x) - G(x)]$

In the same way it can be shown that

$$n^{-1} \sum_{k=1}^n \frac{\partial^2 \varepsilon_k(\alpha)}{\partial \theta^2} \xrightarrow{P} \mathbb{E} \left(\sum_{j=2}^{\infty} j(j-1) \alpha^{j-2} u_{-j} \right) = 0.$$

This finishes the proof of the lemma.

Lemma 4. Assume that the following conditions hold:

$$(i) \quad E(\varepsilon_1)^8 < \infty;$$

$$(ii). \quad g(x) > 0, \quad \lim_{x \rightarrow \infty} g(x) = 0, \quad \sup_x |g'(x)| < \infty$$

Let

$$z_{2n}(x, \tau) := n^{-1/2} \sum_{k=1}^n \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \left[\mathbf{I}(\varepsilon_k(\alpha + n^{-1/2} \tau) \leq x) - \mathbf{I}(\varepsilon_k \leq x) \right].$$

then

$$\left| z_{2n}(x, \tau) + \tau g(x) \frac{1}{1 - \alpha^2} E \varepsilon_1^2 \right| = o_P(1).$$

Proof.

$$z_{2n}(x, \tau) = n^{-1/2} \sum_{k=1}^n \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \left[\mathbf{I} \left(\varepsilon_k \leq x - \tau n^{-1/2} \mu_{k-1} - \sigma_k(\tau) \right) - \mathbf{I}(\varepsilon_k \leq x) \right].$$

consider a process $v_n(x, \tau)$:

$$v_n(x, \tau) := n^{-1} \sum_{k=1}^n \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \left\{ \mathbf{I} \left(\varepsilon_k \leq x - \tau n^{-1/2} \mu_{k-1} - \sigma_k(\tau) \right) - G(x - \tau n^{-1/2} \mu_{k-1} - \sigma_k(\tau)) \right\},$$

then

$$z_{2n} = n^{1/2} [v_n(x, \tau) - \tilde{v}_n(x)] + n^{-1/2} \sum_{k=1}^n \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \left[G(x - \tau n^{-1/2} \mu_{k-1} - \sigma_k(\tau)) - G(x) \right].$$

By analogy to lemma 3 it can be shown that

$$\sup_{x \in \mathbf{R}^1, |\tau| \leq \theta} n^{1/2} [v_n(x, \tau) - \tilde{v}_n(x)] = o_P(1).$$

Consider a process

$$\begin{aligned} n^{-1/2} \sum_{k=1}^n \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \left[G(x - \tau n^{-1/2} \mu_{k-1} - \sigma_k(\tau)) - G(x) \right] &= \\ &= n^{-1/2} \sum_{k=1}^n \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \left(-\tau n^{-1/2} \mu_{k-1} - \sigma_k(\tau) \right) g(x) + \\ &+ n^{-1/2} \sum_{k=1}^n \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \left(-\tau n^{-1/2} \mu_{k-1} - \sigma_k(\tau) \right)^2 g'(\xi) \xrightarrow{P} \\ &\xrightarrow{P} -\tau g(x) \mathbf{E} \left\{ \left(\sum_{j=1}^{\infty} j \alpha^{j-1} u_{k-j} \right) \left(\sum_{j=0}^{\infty} \alpha^j \varepsilon_{k-j-1} \right) \right\}, \end{aligned}$$

where $\xi \in [x - \tau n^{-1/2} \mu_{k-1} - \sigma_k(\tau), x]$, and

$$\mathbf{E} \left\{ \left(\sum_{j=1}^{\infty} j \alpha^{j-1} u_{k-j} \right) \left(\sum_{j=0}^{\infty} \alpha^j \varepsilon_{k-j-1} \right) \right\} = \frac{1}{1 - \alpha^2} \mathbf{E} \varepsilon_1^2,$$

what proves the lemma.

Proof of Theorem 1

Rewrite the process in the following way:

$$n^{1/2}[u_n(x, \alpha + n^{-1/2}\tau) - u_n(x, \alpha)] = z_{1n}(x, \tau) + z_{2n}(x, \tau),$$

where

$$z_{1n}(x, \tau) := n^{-1/2} \sum_{k=1}^n \left(\frac{\partial \varepsilon_k(\alpha + n^{-1/2}\tau)}{\partial \theta} - \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \right) \mathbf{I} \left(\varepsilon_k(\alpha + n^{-1/2}\tau) \leq x \right),$$

$$z_{2n}(x, \tau) := n^{-1/2} \sum_{k=1}^n \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \left[\mathbf{I} \left(\varepsilon_k(\alpha + n^{-1/2}\tau) \leq x \right) - \mathbf{I}(\varepsilon_k \leq x) \right].$$

Now it can be seen that lemma 3 and lemma4 prove the theorem.

Proof of Theorem 2

1)

$$\lambda(\alpha) = - \int_{-\infty}^{\infty} g d\Psi \cdot \mathbf{E} \varepsilon_1^2 \cdot \frac{1}{1 - \alpha^2},$$

$$l_n(\theta) = n^{-1} \sum_{k=1}^n \frac{\partial \varepsilon_k(\theta)}{\partial \theta} \Psi(\varepsilon_k(\theta)), \quad \tilde{l}_n(\alpha) := \frac{1}{n} \sum_{k=1}^n \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \Psi(\varepsilon_k)$$

Without loss of generality we consider $\lambda(\alpha) > 0$, such that there exists $A > 0$ satisfying

$$(2) \quad n^{1/2} l_n(\alpha + n^{-1/2} A) = n^{1/2} \tilde{l}_n(\alpha) + \lambda(\alpha) A + o_P(1) > 0;$$

$$(3) \quad n^{1/2} l_n(\alpha - n^{-1/2} A) = n^{1/2} \tilde{l}_n(\alpha) - \lambda(\alpha) A + o_P(1) < 0.$$

Since $l_n(\theta)$ is continuous on θ in some neighbourhood of α , then because of (2) and (3), a $n^{1/2}$ - consistent solution of (1) exists in interval $(\alpha - n^{-1/2} A; \alpha + n^{-1/2} A)$.

2)

$$n^{1/2} \left[l_n(\alpha + n^{-1/2}\tau) - \tilde{l}_n(\alpha) \right] = - \int_{-\infty}^{\infty} n^{1/2} \left[u_n(x, \alpha + n^{-1/2}\tau) - \tilde{u}_n(x, \alpha) \right] d\Psi(x) +$$

$$+ \left\{ n^{1/2} \left[u_n(x, \alpha + n^{-1/2}\tau) - \tilde{u}_n(x, \alpha) \right] \Psi(x) \right\}_{-\infty}^{+\infty}.$$

Since $\text{Var} \int_{-\infty}^{+\infty} [\Psi(x)] < \infty$ holds, and because of theorem 1 and conditions imposed on $g(x)$, it can be obtained that $\{n^{1/2} [u_n(x, \alpha + n^{-1/2}\tau) - \tilde{u}_n(x, \alpha)] \Psi(x)\}_{-\infty}^{+\infty} = 0$, and also that

$$\begin{aligned} n^{1/2} [l_n(\alpha + n^{-1/2}\tau) - \tilde{l}_n(\alpha)] &= - \int_{-\infty}^{\infty} n^{1/2} [u_n(x, \alpha + n^{-1/2}\tau) - \tilde{u}_n(x, \alpha)] d\Psi(x) = \\ &= \tau \cdot \frac{1}{1 - \alpha^2} \cdot \mathbb{E}\varepsilon_1^2 \cdot \int_{-\infty}^{+\infty} g d\Psi + o_P(1) \end{aligned}$$

uniformly on $|\tau| \leq \Theta$.

Let $\hat{\alpha}_n$ be a $n^{1/2}$ -consistent solution of (1), that is $n^{1/2}(\hat{\alpha}_n - \alpha) = O_P(1)$. Since

$$n^{1/2}l_n(\hat{\alpha}_n) = n^{1/2}\tilde{l}_n(\alpha) + \lambda(\alpha)n^{1/2}(\hat{\alpha}_n - \alpha) + o_P(1) = 0,$$

then

$$n^{1/2}(\hat{\alpha}_n - \alpha) = -[\lambda(\alpha)]^{-1} \cdot n^{1/2}\tilde{l}_n(\alpha) + o_P(1).$$

Consider the process

$$n^{1/2}\tilde{l}_n(\alpha) = n^{-1/2} \sum_{k=1}^n \frac{\partial \varepsilon_k(\theta)}{\partial \theta} \cdot \Psi(\varepsilon_k).$$

If we denote

$$\frac{\partial \tilde{\varepsilon}_k(\alpha)}{\partial \theta} = \sum_{j=1}^{\infty} j \alpha^{j-1} u_{k-j},$$

then

$$\mathbb{E} \left(\frac{\partial \tilde{\varepsilon}_k(\alpha)}{\partial \theta} - \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \right)^2 = \mathbb{E} \left(\sum_{j=k+1}^{\infty} j \alpha^{j-1} u_{k-j} \right)^2 \leq c \alpha_1^k$$

for some α_1 , $1 > \alpha_1 > |\alpha|$.

$$\begin{aligned} &\mathbb{E} \left| n^{-1/2} \sum_{k=1}^n \left(\frac{\partial \tilde{\varepsilon}_k(\alpha)}{\partial \theta} - \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \right) \Psi(\varepsilon_k) \right| \leq \\ &\leq n^{-1/2} \sum_{k=1}^n \left\{ \mathbb{E} \left(\frac{\partial \tilde{\varepsilon}_k(\alpha)}{\partial \theta} - \frac{\partial \varepsilon_k(\alpha)}{\partial \theta} \right)^2 \right\}^{1/2} \cdot \{\mathbb{E}\Psi^2(\varepsilon_k)\}^{1/2} \leq \\ &\leq cn^{-1/2} \sum_{k=1}^n \alpha_1^k \{\mathbb{E}\Psi^2(\varepsilon_k)\}^{1/2} = cn^{-1/2} (\mathbb{E}\Psi^2(\varepsilon_1)) \sum_{k=1}^n \alpha_1^k = o(1). \end{aligned} \quad (*)$$

Since $\xi_k := \frac{\partial \tilde{\varepsilon}_k(\alpha)}{\partial \theta} \Psi(\varepsilon_k)$ is strictly stationary and forms a martingale-difference, then the central limit theorem can be applied to it:

$$n^{-1/2} \sum_{k=1}^n \xi_k \longrightarrow N \left(0, E \left(\sum_{j=1}^{\infty} j \alpha^{j-1} u_{-j} \right)^2 \cdot E \Psi^2(\varepsilon_1) \right).$$

Because of the proved property (*) we obtain

$$n^{1/2} \tilde{l}_n(\alpha) \longrightarrow N \left(0, E \left(\sum_{j=1}^{\infty} j \alpha^{j-1} u_{-j} \right)^2 \cdot E \Psi^2(\varepsilon_1) \right).$$

$$n^{1/2}(\hat{\alpha}_n - \alpha) = -[\lambda(\alpha)]^{-1} \cdot n^{1/2} \tilde{l}_n(\alpha) + o_P(1) \longrightarrow N \left(0, \frac{E \left(\sum_{j=1}^{\infty} j \alpha^{j-1} u_{-j} \right)^2 \cdot E \Psi^2(\varepsilon_1)}{\left(\int_{-\infty}^{+\infty} g d\Psi \right)^2 \cdot \frac{1}{(1-\alpha^2)^2} (E \varepsilon_1^2)^2} \right).$$

Since

$$E \left(\sum_{j=1}^{\infty} j \alpha^{j-1} u_{-j} \right)^2 = \frac{E \varepsilon_1^2}{1 - \alpha^2},$$

then

$$\sigma_{\Psi}^2(\alpha) = (1 - \alpha^2) \frac{E \Psi^2(\varepsilon_1)}{E \varepsilon_1^2 \left(\int_{-\infty}^{+\infty} g d\Psi \right)^2},$$

what proves theorem 2.

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